Module – 3 Lecture Notes – 3

Simplex Method - I

Introduction

It is already stated in a previous lecture that the most popular method used for the solution of *Linear Programming Problems* (LPP) is the *simplex method*. In this lecture, motivation for simplex method will be discussed first. Simplex algorithm and construction of simplex tableau will be discussed later with an example problem.

Motivation for Simplex method

Recall from the second class that the optimal solution of a LPP, if exists, lies at one of the vertices of the feasible region. Thus one way to find the optimal solution is to find all the basic feasible solutions of the canonical form and investigate them one-by-one to get at the optimal. However, again recall the example at the end of the first class that, for 10 equations with 15 variables there exists a huge number (${}^{15}c_{10} = 3003$) of basic feasible solutions. In such a case, inspection of all the solutions one-by-one is not practically feasible. However, this can be overcome by simplex method. Conceptual principle of this method can be easily understood for a three dimensional case (however, simplex method is applicable for any higher dimensional case as well).

Imagine a feasible region (i.e., volume) bounded by several surfaces. Each vertex of this volume, which is a basic feasible solution, is connected to three other adjacent vertices by a straight line to each being the intersection of two surfaces. Being at any one vertex (one of the basic feasible solutions), *simplex algorithm* helps to move to another adjacent vertex which is closest to the optimal solution among all the adjacent vertices. Thus, it follows the shortest route to reach the optimal solution from the starting point. It can be noted that the shortest route consists of a sequence of basic feasible solutions which is generated by *simplex algorithm*. The basic concept of simplex algorithm for a 3-D case is shown in Fig 1.

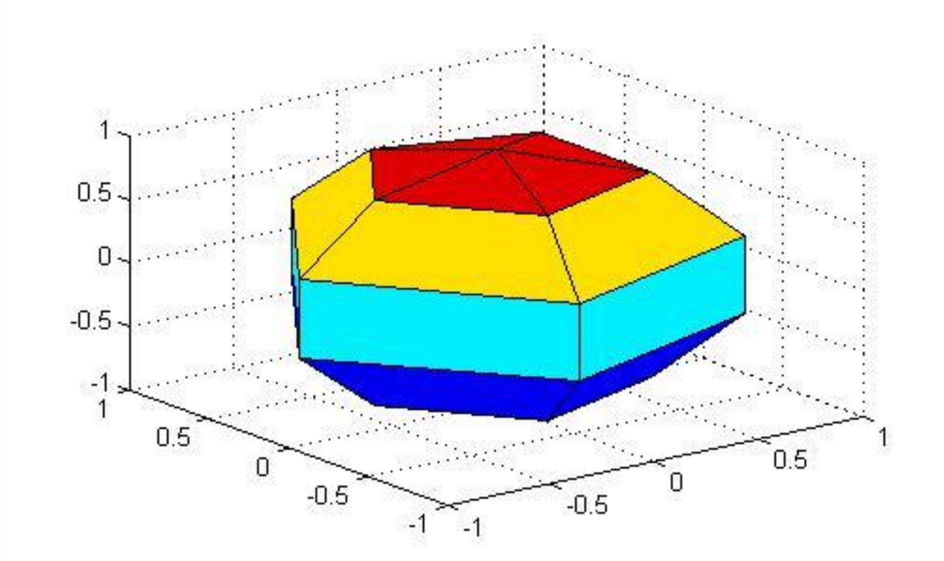

Fig 1.

The general procedure of simplex method is as follows:

1. General form of given LPP is transformed to its *canonical form* (refer Lecture note 1).
2. A basic feasible solution of the LPP is found from the canonical form (there should exist at least one).
3. This initial solution is moved to an adjacent basic feasible solution which is closest to the optimal solution among all other adjacent basic feasible solutions.
4. The procedure is repeated until the optimum solution is achieved.

Step three involves *simplex algorithm* which is discussed in the next section.

Simplex algorithm

Simplex algorithm is discussed using an example of LPP. Let us consider the following problem.

$$
\begin{aligned}
\text{Maximize} \quad & Z = 4x_1 - x_2 + 2x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + 2x_3 \le 6 \\
& x_1 - 4x_2 + 2x_3 \le 0 \\
& 5x_1 - 2x_2 - 2x_3 \le 4 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Simplex algorithm is used to obtain the solution of this problem. First let us transform the LPP to its standard form as shown below.

$$
\begin{array}{ll}
\text{Maximize} & Z = 4x_1 - x_2 + 2x_3 \\
\text{subject to} & 2x_1 + x_2 + 2x_3 + x_4 = 6 \\
 & x_1 - 4x_2 + 2x_3 + x_5 = 0 \\
 & 5x_1 - 2x_2 - 2x_3 + x_6 = 4 \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{array}
$$

It can be recalled that x_4 , x_5 and x_6 are slack variables. Above set of equations, including the objective function can be transformed to canonical form as follows:

$$
\begin{array}{l}
-4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6 + Z = 0 \\
2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 = 6 \\
x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 = 0 \\
5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 = 4
\end{array}
$$

The basic solution of above canonical form is $x_4 = 6$, $x_5 = 0$, $x_6 = 4$, $x_1 = x_2 = x_3 = 0$ and $Z = 0$. It can be noted that, x_4 , x_5 and x_6 are known as basic variables and x_1 , x_2 and x_3 are known as nonbasic variables of the canonical form shown above. Let us denote each equation of above canonical form as:

$$
\begin{array}{ll}
(Z) & -4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6 + Z = 0 \\
(x_4) & 2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 = 6 \\
(x_5) & x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 = 0 \\
(x_6) & 5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 = 4
\end{array}
$$

For the ease of discussion, right hand side constants and the coefficients of the variables are symbolized as follows:

$$
\begin{array}{ll}
(Z) & c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 + c_5 x_5 + c_6 x_5 + Z = b \\
(x_4) & c_{41} x_1 + c_{42} x_2 + c_{43} x_3 + c_{44} x_4 + c_{45} x_5 + c_{46} x_5 = b_4 \\
(x_5) & c_{51} x_1 + c_{52} x_2 + c_{53} x_3 + c_{54} x_4 + c_{55} x_5 + c_{56} x_5 = b_5 \\
(x_6) & c_{61} x_1 + c_{62} x_2 + c_{63} x_3 + c_{64} x_4 + c_{65} x_5 + c_{66} x_5 = b_6
\end{array}
$$

The left-most column is known as *basis* as this is consisting of basic variables. The coefficients in the first row ($c_1 \wedge c_6$) are known as *cost coefficients*. Other subscript notations are self explanatory and used for the ease of discussion. For each coefficient, first subscript indicates the subscript of the basic variable in that equation. Second subscript indicates the subscript of variable with which the coefficient is associated. For example, c_{52} is the coefficient of x_2 in the equation having the basic variable x_5 with nonzero coefficient (i.e., c_{55} is nonzero).

This completes first step of calculation. After completing each step (iteration) of calculation, three points are to be examined:

1. Is there any possibility of further improvement?
2. Which nonbasic variable is to be entered into the basis?
3. Which basic variable is to be exited from the basis?

The procedure to check these points is discussed next.

1. **Is there any possibility of further improvement?**

   If any of the cost coefficients is negative, further improvement is possible. In other words, if all the cost coefficients are nonnegative, the basic feasible solution obtained in that step is optimum.

2. **Which nonbasic variable is to be entered?**

   Entering nonbasic variable is decided such that the unit change of this variable should have maximum effect on the objective function. Thus the variable having the coefficient which is minimum among all the cost coefficients is to be entered, i.e., x_S is to be entered if cost coefficient c_S is minimum.

3. **Which basic variable is to be exited?**

   After deciding the entering variable x_S , x_r (from the set of basic variables) is decided to be the exiting variable if $\frac{b_r}{c_{rs}}$ is minimum for all possible *r*, provided c_{rs} is positive.

It can be noted that, c_{rs} is considered as pivotal element to obtain the next canonical form.

In this example, $c_1\ (= -4)$ is the minimum. Thus, x_1 is the entering variable for the next step of calculation. r may take any value from 4, 5 and 6. It is found that $\frac{b_4}{c_{41}} = \frac{6}{2} = 3$, $\frac{b_5}{c_{51}} = \frac{0}{1} = 0$ and $\frac{b_6}{c_{61}} = \frac{4}{5} = 0.8$. As, $\frac{b_5}{c_{51}}$ is minimum, r is 5. Thus x_5 is to be exited and c_{51} is the pivotal element and x_5 is replaced by x_1 in the basis. Set of equations are transformed through pivotal operation to another canonical form considering c_{51} as the pivotal element. The procedure of pivotal operation is already explained in first class. However, as a refresher it is explained here once again.

1. Pivotal row is transformed by dividing it with the pivotal element. In this case, pivotal element is 1.
2. For other rows: Let the coefficient of the element in the pivotal column of a particular row be "*l*". Let the pivotal element be "*m*". Then the pivotal row is multiplied by $l\ /\ m$ and then subtracted from that row to be transformed. This operation ensures that the coefficients of the element in the pivotal column of that row becomes zero, e.g., *Z row*: l = -4 , m = 1. So, pivotal row is multiplied by l / m = -4 / 1 = -4, obtaining

$$-4x_1 + 16x_2 - 8x_3 + 0x_4 - 4x_5 + 0x_6 = 0$$

This is subtracted from *Z row* obtaining,

$$0x_1 - 15x_2 + 6x_3 + 0x_4 + 4x_5 + 0x_6 + Z = 0$$

The other two rows are also suitably transformed.

After the pivotal operation, the canonical form obtained is shown below.

$$\begin{array}{ll}
(Z) & 0x_1 - 15x_2 + 6x_3 + 0x_4 + 4x_5 + 0x_6 + Z = 0 \\
(x_4) & 0x_1 + 9x_2 - 2x_3 + 1x_4 - 2x_5 + 0x_6 = 6 \\
(x_1) & 1x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 = 0 \\
(x_6) & 0x_1 + 18x_2 - 12x_3 - 0x_4 - 5x_5 + 1x_6 = 4
\end{array}$$

The basic solution of above canonical form is $x_1 = 0$, $x_4 = 6$, $x_6 = 4$, $x_3 = x_4 = x_5 = 0$ and $Z = 0$. However, this is not the optimum solution as the cost coefficient c_2 is negative. It is observed that c_2 (= -15) is minimum. Thus, s = 2 and x_2 is the entering variable. r may take any value from 4, 1 and 6. However, c_{12} $(=-4)$ is negative. Thus, r may be either 4 or 6. It is found that, $\frac{b_4}{c_{42}} = \frac{6}{9} = 0.667$, and $\frac{b_6}{c_{62}} = \frac{4}{18} = 0.222$. As $\frac{b_6}{c_{62}}$ is minimum, r is 6 and x_6 is to be exited from the basis. c_{62} (=18) is to be treated as pivotal element. The canonical form for next iteration is as follows:

$$
\begin{array}{llr}
(Z) & 0x_1 + 0x_2 - 4x_3 + 0x_4 - \frac{1}{6}x_5 + \frac{5}{6}x_6 + Z & = \frac{10}{3} \\
(x_4) & 0x_1 + 0x_2 + 4x_3 + 1x_4 + \frac{1}{2}x_5 - \frac{1}{2}x_6 & = 4 \\
(x_1) & 1x_1 + 0x_2 - \frac{2}{3}x_3 + 0x_4 - \frac{1}{9}x_5 + \frac{2}{9}x_6 & = \frac{8}{9} \\
(x_2) & 0x_1 + 1x_2 - \frac{2}{3}x_3 + 0x_4 - \frac{5}{18}x_5 + \frac{1}{18}x_6 & = \frac{2}{9}
\end{array}
$$

The basic solution of above canonical form is $x_1 = \frac{8}{9}$, $x_2 = \frac{2}{9}$, $x_4 = 4$, $x_2 = x_3 = x_5 = 0$ and $Z = \frac{10}{3}$.

It is observed that c_3 (= -4) is negative. Thus, optimum is not yet achieved. Following similar procedure as above, it is decided that x_3 should be entered in the basis and x_4 should be exited from the basis. Thus, x_4 is replaced by x_3 in the basis. Set of equations are transformed to another canonical form considering c_{43} (= 4) as pivotal element. By doing so, the canonical form is shown below.

$$
\begin{array}{llr}
(Z) & 0x_1 + 0x_2 + 0x_3 + 1x_4 + \frac{1}{3}x_5 + \frac{1}{3}x_6 + Z & = \frac{22}{3} \\
(x_3) & 0x_1 + 0x_2 + 1x_3 + \frac{1}{4}x_4 + \frac{1}{8}x_5 - \frac{1}{8}x_6 & = 1 \\
(x_1) & 1x_1 + 0x_2 + 0x_3 + \frac{1}{6}x_4 - \frac{1}{36}x_5 + \frac{5}{36}x_6 & = \frac{14}{9} \\
(x_2) & 0x_1 + 1x_2 + 0x_3 + \frac{1}{6}x_4 - \frac{7}{36}x_5 - \frac{1}{36}x_6 & = \frac{8}{9}
\end{array}
$$

The basic solution of above canonical form is $x_1 = \frac{14}{9}, x_2 = \frac{8}{9}, x_3 = 1, x_4 = x_5 = x_6 = 0$ and $Z = \frac{22}{3}$.

It is observed that all the cost coefficients are positive. Thus, optimum is achieved. Hence, the optimum solution is

$$Z = \frac{22}{3} = 7.333$$

$$x_2 = \frac{8}{9} = 0.889$$

The calculation shown above can be presented in a tabular form, which is known as *Simplex Tableau*. Construction of *Simplex Tableau* will be discussed next.

Construction of Simplex Tableau

Same LPP is considered for the construction of *simplex tableau*. This helps to compare the calculation shown above and the construction of *simplex tableau* for it.

After preparing the canonical form of the given LPP, simplex tableau is constructed as follows.

Iteration	Basis	Z	x_1	x_2	x_3	x_4	x_5	x_6	b_r	$\frac{b_r}{c_{rs}}$
			Variables							
1	Z	1	-4	1	-2	0	0	0	0	--
	x_4	0	2	1	2	1	0	0	6	3
	x_5	0	1	-4	2	0	1	0	0	0
	x_6	0	5	-2	-2	0	0	1	4	$\frac{4}{5}$

Pivotal Row

Pivotal Column

Pivotal Element

After completing each iteration, the steps given below are to be followed. Logically, these steps are exactly similar to the procedure described earlier. However, steps described here are somewhat mechanical and easy to remember!

Check for optimum solution:

1. Investigate whether all the elements in the first row (i.e., Z row) are nonnegative or not. Basically these elements are the coefficients of the variables headed by that column. If all such coefficients are nonnegative, optimum solution is obtained and no need of further iterations. If any element in this row is negative, the operation to obtain simplex tableau for the next iteration is as follows:

Operations to obtain next simplex tableau:

2. The entering variable is identified (described earlier). The corresponding column is marked as *Pivotal Column* as shown above.
3. The exiting variable from the basis is identified (described earlier). The corresponding row is marked as *Pivotal Row* as shown above.
4. Coefficient at the intersection of *Pivotal Row* and *Pivotal Column* is marked as *Pivotal Element* as shown above.
5. In the basis, the exiting variable is replaced by entering variable.

6. All the elements in the pivotal row are divided by pivotal element.
7. For any other row, an elementary operation is identified such that the coefficient in the pivotal column in that row becomes zero. The same operation is applied for all other elements in that row and the coefficients are changed accordingly. A similar procedure is followed for all other rows.

   For example, say, (2 x *pivotal element* + *pivotal coefficient in first row*) produce zero in the pivotal column in first row. The same operation is applied for all other elements in the first row and the coefficients are changed accordingly.

Simplex tableaus for successive iterations are shown below. *Pivotal Row*, *Pivotal Column* and *Pivotal Element* for each tableau are marked as earlier for the ease of understanding.

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	x_5	x_6		
2	Z	1	0	-15	6	0	4	0	0	--
	x_4	0	0	9	-2	1	-2	0	6	1/3
	x_1	0	1	-4	2	0	1	0	0	--
	x_6	0	0	18	-12	0	-5	1	4	2/9

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Iteration	Basis	Z	x_1	x_2	x_3	x_4	x_5	x_6	b_r	$\frac{b_r}{c_{rs}}$
3	Z	1	0	0	-4	0	$-\frac{1}{6}$	$\frac{5}{6}$	$\frac{10}{3}$	--
	x_4	0	0	0	4	1	$\frac{1}{2}$	$-\frac{1}{2}$	4	1
	x_1	0	1	0	$-\frac{2}{3}$	0	$-\frac{1}{9}$	$\frac{2}{9}$	$\frac{8}{9}$	--
	x_2	0	0	1	$-\frac{2}{3}$	0	$-\frac{5}{18}$	$\frac{1}{18}$	$\frac{2}{9}$	--
4	Z	1	0	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{22}{3}$	
	x_3	0	0	0	1	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{8}$	1	
	x_1	0	1	0	0	$\frac{1}{6}$	$-\frac{1}{36}$	$\frac{2}{9}$	$\frac{14}{9}$	
	x_2	0	0	1	0	$\frac{1}{6}$	$-\frac{7}{36}$	$-\frac{1}{36}$	$\frac{8}{9}$	

Optimum value of Z

All the coefficients a
nonnegative. Thus o
solution is achieved.

solution is achieved. Optimum value of Z is

As all the elements in the first row (i.e., Z row), at iteration 4, are nonnegative, optimum $\frac{22}{3} = 7.333$ as shown above. Corresponding

values of basic variables are $x_1 = \frac{14}{9} = 1.556$, $x_2 = \frac{8}{9} = 0.889$, $x_3 = 1$ and those of nonbasic variables are all zero (i.e., $x_4 = x_5 = x_6 = 0$).

It can be noted that at any iteration the following two points must be satisfied:

1. All the basic variables (other than Z) have a coefficient of zero in the Z row.
2. Coefficients of basic variables in other rows constitute a unit matrix.

If any of these points are violated at any iteration, it indicates a wrong calculation. However, reverse is not true.